# BACKPAPER: ALGEBRA II

Date:

The Total points is **100**.

Notation: $\mathbb{F}_p$ denotes the finite field of $p$ elements where $p$ is a prime number.

(1) (10 points) Let $L/\mathbb{Q}$ be a finite extension. Show that $L$ contains finitely many roots of unity.

(2) (20 points) Let $\alpha$ be algebraic over $\mathbb{Q}$, $L$ be the Galois closure of $\mathbb{Q}(\alpha)/\mathbb{Q}$ and $G = \mathrm{Gal}(L/\mathbb{Q})$. For any prime $p$ dividing the order of $G$, show that there exists a subfield $F \subset L$ such that $[L : F] = p$ and $L = F(\alpha)$.

(3) (15 points) Let $F$ be a field of characteristic $p > 0$. Show that $K/F$ is purely inseparable iff there is a unique $F$-embedding of $K$ into the algebraic closure of $F$.

(4) (20 points) Show that $\mathrm{Gal}(\bar{F}/F) = \hat{\mathbb{Z}}$ where $F$ is a finite field and $\bar{F}$ its algebraic closure.

(5) (15 points) Let $L/F$ be a finite field extension. Show that if $L/F$ is a Galois extension then there exist an $\alpha \in L$ such that its trace $T_{L/F}(\alpha) \neq 0$.

(6) (20 points) Let $F$ be a field of characteristic $p$ and $a \in F$ be such that the polynomial $f(Z) = Z^p - Z - a$ is irreducible in $F[Z]$. Let $\alpha$ be a root of $f(Z)$ in some fixed algebraic closure $\bar{F}$ of $F$. Let $\beta \in \bar{F}$ be such that $c := \beta^p - \beta \in F$. Show that if $a - uc = x^p - x$ for some nonzero $u \in \mathbb{F}_p$ and some $x \in F$, then $F[\alpha] = F[\beta]$.